

Likelihood inference for continuous-time state-space models

*Inferenza basata sulla verosimiglianza
per modelli stato-spazio a tempo continuo*

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Riassunto: In questo lavoro consideriamo un processo di Poisson doppiamente stocastico, che può essere interpretato come un modello stato-spazio non lineare a tempo continuo. Abbiamo sviluppato un algoritmo sequenziale Monte Carlo che rende possibile il filtraggio del processo latente e il calcolo puntuale della verosimiglianza. L'errore dell'approssimazione Monte Carlo e il tempo di calcolo sono stati ridotti applicando il teorema di Rao-Blackwell. Lo stesso algoritmo è stato utilizzato per la stima dei parametri del modello. Illustriamo una applicazione a dati sismologici.

Keywords: double stochastic Poisson process, pure jump Markov process, filter, smoother.

1. Introduction

The development of new inferential methodologies for the analysis of continuous-time nonlinear state-space models is extremely interesting in many application fields.

In particular we employed these models for the study of seismic sequences. The temporal variations of the geophysical and geological features of a seismic region could produce changes in the physical system in which different occurrence rates of the earthquakes are observed. For example, after a period of quiescence an increasing rate of released stress could be observed followed by a strong earthquake and a period of aftershocks.

In this paper we propose a state-space model aimed to describe both the transitions between different states of the physical system and the inner dynamics of each state (Varini, 2005). A state-space model is composed by two processes: the state process $\mathbf{X} = \{X_t\}_{t \geq 0}$, recording the transitions from one state to another, and the observation process $\mathbf{Y} = \{Y_t\}_{t \geq 0}$, recording the earthquake occurrences. We chose the state process to be a homogeneous pure jump Markov process having only three possible states, respectively denoted by 1, 2, 3. The observation process is defined by a point process having a left-continuous conditional intensity function $\lambda(j, t)$ when $X_t = j$. The expressions of $\lambda(j, t)$ for $j = 1, 2, 3$ are chosen from the main point processes for seismic sequences proposed in the literature (Ogata, 1988; Zheng and Vere-Jones, 1994). The proposed state-space model is a version of a doubly stochastic Poisson process. Let $t_1 < t_2 < \dots < t_N$ denote the time occurrences of the earthquakes in a interval $[0, t_{N+1}]$ and let $y_{0:t}, x_{0:t}$ denote respectively the observations and the sequence of the states in $[0, t]$.

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2. Filter and smoother

By using the innovation method and the Feynman-Kac formula (Brémaud, 1981; Gerardi and Tardelli, 2001), we determine the following iterative system of stochastic differential equations for each time $t \geq 0$ and state $j = 1, 2, 3$:

$$\begin{cases} \varphi_0(j) = P(X_0 = j) \\ \varphi_t(j) = E \left(I(X_t^* = j) \exp \left\{ - \int_{t_n}^t \sum_{i \in \mathcal{S}} \lambda(i, s) I(X_s^* = i) ds \right\} \right), t \in (t_n, t_{n+1}) \\ \varphi_{t_{n+1}}(j) = \lambda(j, t_{n+1}) \varphi_{t_{n+1}^-}(j) \left(\sum_{k=1}^3 \varphi_{t_{n+1}^-}(k) \right)^{-1} \end{cases}$$

The filtering probabilities and the likelihood function are respectively given by

$$P(x_t = j \mid y_{0:t}) = \frac{\varphi_t(j)}{\sum_{k=1}^3 \varphi_t(k)} \quad , \quad \mathcal{L} = \prod_{n=1}^{N+1} \sum_{k=1}^3 \varphi_{t_n}(k) \cdot \prod_{n=1}^N \sum_{k=1}^3 \varphi_{t_n^-}(k)$$

where $I(\cdot)$ denotes the indicator function and $E(\cdot)$ the expectation with respect to the law of a process $\{X_t^*\}_{t \geq 0}$ defined in (t_n, t_{n+1}) having the same generator of the state process $\mathbf{X}$ and initial distribution $P(x_{t_n} = j \mid y_{0:t_n})$ for $j = 1, 2, 3$.

The filtering probabilities can be sequentially approximated by Monte Carlo simulations of $\{X_t^*\}$ in each sub-interval. The Monte Carlo error and the time of computations are drastically reduced by decomposing the expectation in the sum of three conditional expectations respectively obtained by assuming that zero, one or more than one state changes occur in the considered subinterval; this Rao-Blackwellisation technique allows exact or numerical evaluation of the first two conditional expectations, while just the third needs to be approximated by Monte Carlo simulation.

It is possible to implement a quick backward smoother for the approximations of $P(x_t = j \mid y_{0:t_{N+1}})$ at each time t and state j . This smoother is a special and extended version to the continuous-time case of the smoother proposed by Godsill *et al.* (2004).

The proposed filter and smoother can be integrated into some inferential procedures (maximum likelihood estimation, Monte Carlo EM algorithm or Bayesian estimation methods) for the estimation of the unknown model parameters.

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